NOTATION

A, B, linear operators; u, element of solution space U; $\bar{f}$, exact reference data; $\tilde{f}$, reference data uncertainty; δ, value of reference data uncertainty; A^{-1}, inverse operator; $u^{(k)}(\tau)$, k-th derivative of function u; τ_m, length of observation interval; $\varphi_i(t)$, polynomials of degree i − 1; A^*, B^*, L^*, operators conjugate to the operators A, $\bar{B}$, L; $J'g$, discrepancy functional gradient; β_n, descent step along the discrepancy antigradient for the n-th iteration; $K(\tau - \xi)$, kernel of integral equation; $q(\tau)$, heat flux; $T_\delta(\tau)$, measured temperature inside body.

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OPTIMAL CHOICE OF DESCENT STEPS IN GRADIENT METHODS OF SOLUTION OF INVERSE HEAT-CONDUCTION PROBLEMS

E. A. Artyukhin and S. V. Rumyantsev

UDC 536.24.02

Modifications are proposed for the methods of steepest descent and conjugate gradients for the solution of multiparameter inverse problems in heat conduction.

In the solution of inverse heat-conduction problems it often becomes necessary to determine several independent functions and parameters at once. Such multiparameter problems arise in the solution of coefficient-type inverse problems, in the joint determination of the external thermal load and some thermophysical characteristics of the body, etc. An attempt to take the most complete account of the a priori information about the desired solution may also lead to such problems.

In the solution of boundary-value inverse problems with one unknown (a function or a parameter) it has been found very effective to use algorithms based on gradient methods of minimization [1-3]. The use of these methods in a case when it is necessary to determine several independent variables is made more difficult by the fact that the descent step is chosen to be the same for all components of the direction of descent. Such a method of choosing the step frequently leads to very slow convergence of the gradient methods. The convergence may be speeded up considerably by choosing different descent steps for the different components of the gradient of the minimizing functional, i.e., to determine not one step but a vector of steps from the condition that the target functional has a minimum with respect to this vector at each iteration.

We shall show how this method can be used for constructing gradient algorithms for the solution of boundary-value inverse problems in heat conduction when a priori information concerning the smoothness of the desired solution is available.

A boundary-value inverse heat-conduction problem for bodies with constant thermophysical characteristics can be reduced to the solution of the first-order equation

$$Au = f_\delta, \quad u \in U, \quad f_\delta \in F, \tag{1}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 2, pp. 264-269, August, 1980. Original article submitted January 2, 1980.

where A is a linear continuous operator realizing the transformation $U = L_2[0, \tau_m] \to L_2[0, \tau_m] = F$ and specifying the variation of the thermal regime within the body as a function of the unknown boundary condition $f_\delta = \bar{f} + \tilde{f}$, where $\bar{f}$ is the exact right side (e.g., the temperature at some point in the interior of the body) and $\tilde{f}$ is the random noise in the measurement of $\bar{f}$ and $\|\tilde{f}\|_{L_2} = \delta$; τ_m is the length of the interval of observation. The inverse operator is usually unbounded.

In [4] we proposed a method for taking account of information concerning the smoothness of the desired function when it is known that the solution $u(\tau)$ is continuous, together with all its derivatives up to some order k inclusive. We take account of the smoothness because the solution is sought in the transform of the integral operator L, i.e., in the form

$$\bar{u} = L\bar{g} = \int_{t_1}^{\tau} \dots \int_{t_k}^{t} \bar{\vartheta}(\xi)\, d\xi dt + \dots + \sum_{i=1}^{k} \bar{c}_i \varphi_i = L_0\bar{\vartheta} + \sum_{i=1}^{k} \bar{c}_i \varphi_i,$$

where $\bar{g} = \{\bar{\vartheta}, \bar{c}_1, \dots, \bar{c}_k\}$, $\bar{\vartheta}(\tau) = u^{(k)}(\tau)$, $t_i \in [0, \tau_m]$; $\varphi_i(\tau)$ are polynomials of degree $(i-1)$; $\bar{c}_i$ are the values of $\bar{u}(\tau)$ and its derivatives at the points t_i [4].

Using such an approach, we replace the initial problem with the problem of determining the vector $\bar{g}$ belonging to the space $G = L_2[0, \tau_m] \times R^k$, which has the norm $\|g\|^2 = \|\vartheta\|^2_{L_2} + \sum_{i=1}^{k} c_i^2$. Thus, instead of Eq. (1) we must solve the equation

$$Bg = f_\delta, \quad B = AL. \tag{2}$$

We introduce the discrepancy functional

$$J = \frac{1}{2} \|Bg - f_\delta\|^2_{L_2}. \tag{3}$$

The expression for the gradient of the discrepancy functional of Eq. (3) was obtained in [4],

$$\begin{gathered} J'g = \{(L_k^* \dots L_1^* A^* (Au - f_\delta), \quad (\varphi_1, A^*(Au - f_\delta))_{L_2}, \dots, \\ (\varphi_k, A^*(Au - f_\delta))_{L_2}\} = \{(J'g)_0, (J'g)_1, \dots, (J'g)_k\}, \end{gathered} \tag{4}$$

where

$$L_i^* z = \begin{cases} \int_{\tau}^{\tau_m} z(t)\, dt, & \tau > t_i; \\ \int_{0}^{\tau} z(t)\, dt, & \tau \leqslant t_i. \end{cases}$$

Now we consider a gradient algorithm for minimizing the functional (3) in which we make a choice of the descent steps separately for each coordinate of the vector J'g and we construct the process of successive approximation by the formula

$$g_{n+1} = g_n - \sum_{j=0}^{k} \beta_n^j (J'g_n)_j. \tag{5}$$

This approach is a modification of the method of steepest descent. The vector of steps $\beta_n = \{\beta_n^0, \beta_n^1, \dots, \beta_n^k\}$ will be chosen from the condition that we must have a minimum of the discrepancy functional

$$\frac{1}{2} \|Bg_{n+1} - f_\delta\|^2_{L_2} = \min_\beta \frac{1}{2} \|B(g_n - \beta J'g_n) - f_\delta\|^2_{L_2}. \tag{6}$$

Applying the operator L to both parts of the sequence (5), we can obtain a sequence of approximations and a condition for the choice of the vector of steps (6) in terms of the initial problem (1):

$$u_{n+1} = u_n - \sum_{i=1}^{k} \beta_n^i (J'g_n)_i \varphi_i - \beta_n^0 L_0 (J'g_n)_0, \tag{7}$$

$$\frac{1}{2} \|Au_{n+1} - f_\delta\|^2_{L_2} = \min_\beta \frac{1}{2} \|Au_n - AL\beta J'g_n - f_\delta\|^2_{L_2} = \min_\beta J(u_{n+1}). \tag{8}$$

Setting the derivative of $J(u_{n+1})$ with respect to β_n^i equal to zero, we obtain

$$\frac{\partial J(u_{n+1})}{\partial \beta_n^i} = \frac{\partial}{\partial \beta_n^i} \frac{1}{2} \left\| Au_n - f_\delta - \sum_{j=1}^{k} \beta_n^j (J'g_n)_j A\varphi_j - \beta_n^0 AL_0 (J'g_n)_0 \right\|^2_{L_2} = -(Au_n - f_\delta - \beta_n^0 AL_0 (J'g_n)_0,$$

$$(J'g_n)_i\, A\varphi_i)_{L_2} + ((J'g_n)_i\, A\varphi_i,\ \sum_{j=1}^{k} \beta_n^j (J'g_n)_j\, A\varphi_j)_{L_2} = 0, \quad i = 1, \ldots, k,$$

$$\frac{\partial J(u_{n+1})}{\partial \beta_n^0} = -\ (Au_n - f_\delta - \sum_{j=1}^{k} \beta_n^j (J'g_n)_j\, A\varphi_j,$$

$$AL_0 (J'g_n)_0)_{L_2} + \beta_n^0 \|AL_0 (J'g_n)_0\|_{L_2}^2 = 0.$$

From this we obtain a system of equations for determining β_n:

$$\sum_{j=1}^{k} \beta_n^j (J'g_n)_j (A\varphi_i,\ A\varphi_j)_{L_2} + \beta_n^0 (AL_0 (J'g_n)_0,\ A\varphi_i)_{L_2} =$$

$$= (\varphi_i,\ A^* (Au_n - f_\delta))_{L_2} = (J'g_n)_i, \quad i = 1, \ldots, k, \tag{9}$$

$$\sum_{j=1}^{k} \beta_n^k (J'g_n)_j (A\varphi_j,\ AL_0 (J'g_n)_0)_{L_2} + \beta_n^0 \|AL_0 (J'g_n)\|_{L_2}^2 = (A^* (Au_n - f_\delta),$$

$$L_0 (J'g_n)_0)_{L_2} = \|(J'g_n)_0\|_{L_2}^2 .$$

The last equations in the relations (9) follow immediately from the expression for the gradient of J′g.

Thus, to obtain β_n at each iteration we must solve the system of linear algebraic equations (9).

The system of equations (9) can be considerably simplified if we transform the system of functions $\{\varphi_i\}$ into the system $\{\bar{\varphi}_i\}$, so that $(A\bar{\varphi}_i,\ A\bar{\varphi}_j)_{L_2} = \delta_{ij}$. Such a transformation can be carried out easily by applying the Hilbert–Schmidt orthogonalization process to the system $\{A\varphi_i\}$. This is entirely admissible, since the initial system $\{\varphi_i\}$ is linearly independent (φ_i are polynomials of degree i − 1) and the heat-conduction equation with a boundary condition in the form of any polynomial of degree k has a nonzero solution. Consequently the system of functions $\{A\varphi_i\}$ will also be linearly independent. The required system of functions $\{\bar{\varphi}_i\}$ is obtained from the system $\{\varphi_i\}$ by the formulas:

$$\bar{\varphi}_1 = \frac{\varphi_1}{\|A\varphi_1\|_{L_2}}, \quad \bar{\varphi}_i = \frac{\varphi_i - \sum_{l=1}^{i-1} (A\varphi_i,\ A\bar{\varphi}_l)\, \bar{\varphi}_l}{\left\| A\varphi_i - \sum_{l=1}^{i-1} (A\varphi_i,\ A\bar{\varphi}_l)\, A\varphi_l \right\|_{L_2}} .$$

Obviously $(A\bar{\varphi}_i,\ A\bar{\varphi}_j)_{L_2} = \delta_{ij}$, where δ_{ij} is the Kronecker delta. This transformation can be carried out once before beginning the solution of the inverse problem.

As we have already noted when we specified the operator L, the functions φ_i are chosen rather arbitrarily, and therefore, when we pass to the functions $\bar{\varphi}_i$, all the arguments given earlier remain valid.

For the transformed system of functions $\{\bar{\varphi}_i\}$ the system of equations (9) takes the following form

$$(J'g_n)_i\, \beta_n^i + \beta_n^0 (AL_0 (J'g_n)_0,\ A\bar{\varphi}_i)_{L_2} = (J'g_n)_i,$$

$$i = 1, \ldots, k, \tag{10}$$

$$\sum_{j=1}^{k} \beta_n^j (J'g_n)_j (A\bar{\varphi}_j,\ AL_0 (J'g_n)_0)_{L_2} + \beta_n^0 \|AL_0 (J'g_n)_0\|_{L_2}^2 = \|(J'g_n)_0\|_{L_2}^2 .$$

Using the first k equations of the system (10), we can eliminate from the last equation all the β_n^i except β_n^0:

$$\sum_{j=1}^{k} [(Jg_n)_j - \beta_n^0 (A\bar{\varphi}_j,\ AL_0 (J'g_n)_0)_{L_2}] (A\bar{\varphi}_j,\ AL_0 (J'g_n)_0)_{L_2} + \beta_n^0 \|AL_0 (J'g_n)_0\|_{L_2}^2 = \|(J'g_n)_0\|_{L_2}^2 .$$

From this we finally obtain

$$\beta_n^0 = \frac{\|(J'g_n)_0\|_{L_2}^2 - \sum_{j=1}^{k} (J'g_n)_j (A\bar{\varphi}_j,\ AL_0 (J'g_n)_0)_{L_2}}{\|AL_0 (J'g_n)_0\|_{L_2}^2 - \sum_{j=1}^{k} (A\bar{\varphi}_j,\ AL_0 (J'g_n)_0)_{L_2}^2},$$

$$\beta_n^i = 1 - \beta_n^0 \frac{(A\bar{\varphi}_i,\ AL_0(J'g_n)_0)_{L_2}}{(J'g_n)_i}.$$

The calculation of the gradient g_n and the following approximation u_{n+1} is carried out by using functions $\bar{\varphi}_i$.

The method of conjugate gradients can be modified in an analogous manner. Since the descent steps are chosen individually for each component of the vector $J'g_n$, it follows that a correction to the direction of the descent can be made only for the first component. Therefore, the modified method of conjugate gradients can be written as follows:

$$u_{n+1} = u_n - \sum_{j=1}^{k} \beta_n^j (J'g_n)_j \bar{\varphi}_j - \beta_n^0 L_0 (J'g_n)_0 - \gamma L_0 p_{n-1}, \tag{11}$$

where $p_{-1}(\tau) \equiv 0$; $p_0 = \beta_0^0 (J'g_n)_0$; $p_n = \beta_n^0 (J'g_n)_0 + \gamma_n p_{n-1}$; β_n and γ_n are chosen from the condition

$$J(u_{n+1}) = \min_{\beta,\gamma} \frac{1}{2} \left\| Au_n - f_\delta - \sum_{j=1}^{k} \beta^j (J'g_n)_j A\bar{\varphi}_j - \beta^0 AL_0 (J'g_n)_0 - \gamma AL_0 p_{n-1} \right\|_{L_2}^2. \tag{12}$$

Setting the derivatives $J(u_{n+1})$ with respect to β_n^j and γ_n equal to zero and taking account of the fact that $(A\bar{\varphi}_i,\ A\bar{\varphi}_j)_{L_2} = \delta_{ij}$, we obtain

$$\begin{aligned}
&(J'g_n)_i \beta_n^i + \beta_n^0 (A\bar{\varphi}_i,\ AL_0(J'g_n)_0)_{L_2} + \gamma_n (A\bar{\varphi}_i,\ AL_0 p_{n-1})_{L_2} = (J'g_n)_i,\\
&\qquad i = 1, \ldots, k;\\
&\sum_{j=1}^{k} \beta_n^j (J'g_n)_j (A\bar{\varphi}_j,\ AL_0(J'g_n)_0)_{L_2} + \beta_n^0 (AL_0(J'g_n)_0,\\
&AL_0 p_{n-1})_{L_2} + \gamma_n (AL_0(J'g_n)_0,\ AL_0 p_{n-1})_{L_2} + \|(J'g_n)_0\|_{L_2}^2,\\
&\sum_{j=1}^{k} \beta_n^j (J'g_n)_j (A\bar{\varphi}_j,\ AL_0 p_{n-1})_{L_2} + \beta_n^0 (AL_0(J'g_n)_0,\\
&AL_0\ p_{n-1})_{L_2} + \gamma_n \|AL_0 p_{n-1}\|_{L_2}^2 = (p_{n-1},\ (J'g_n)_0)_{L_2}.
\end{aligned} \tag{13}$$

From the system of equations (13) we readily obtain formulas for β_n, γ_n.

As in [1-3], the iterative process can be halted according to the discrepancy principle, i.e., on the basis of the condition $2J(u_n) \simeq \delta^2$.

The proposed modifications of the methods of steepest descent and conjugate gradients may considerably increase the rate of convergence with only small increases in the machine time used for each iteration. This conclusion is completely confirmed by the results of calculations made for simulated examples. Some of these are given in Fig. 1.

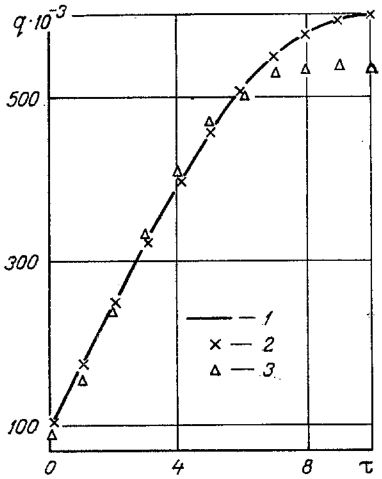

Fig. 1. Reconstruction of the heat flux density q, 10^6 W/m^2: 1) exact solution; 2) reconstructed heat-flux density from exact data; 3) the same for perturbed data; τ = time, sec.

We considered the problem of reconstructing the heat-flux density on one of the boundaries of an unbounded plate in which the process of heat transfer is described by a linear homogeneous heat-conduction equation. The boundary condition of the second kind at the other boundary of the plate was known. As the input data, we used the variation of temperature as a function of time at an interior point of the plate. The solution of the initial heat-conduction problem was represented in integral form. Furthermore, using the method of [1], we passed to a system of linear algebraic equations, for which we constructed the above-described algorithms. The inverse problem was solved both for the exact input data and for input data perturbed by means of a random-number device. When the input-temperature perturbations were up to 10% of the maximum value, a halt by the discrepancy principle was obtained within three to eight iterations, depending on the variant involved. When a constant descent parameter was used in analogous simulated examples, 30 to 60 iterations were required.

NOTATION

A, B, L, linear operators; u, element of the solution space U; $\bar{f}$, exact initial data; $\tilde{f}$, error in the initial data; δ, value of the error in the initial data; A^{-1}, inverse operator; $u^{(k)}(\tau)$, the k-th derivative of the function u; $\varphi_i(\tau)$, polynomials of degree i − 1; A*, B*, L*, operators conjugate to the operators A, B, L; J(g), discrepancy functional; J′g, gradient of the discrepancy functional; β_n^i, depth of descent with respect to the i-th component of the antigradient of the discrepancy in the n-th iteration; τ_m, length of the observation interval.

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REGULARIZING ALGORITHM FOR INVERTING THE ABEL EQUATION

Yu. E. Voskoboinikov

UDC 517.948.32

The article presents a regularizing algorithm for solving the Abel equation using information on the statistics of the error of measurement of the right-hand side of the equation.

Optical methods have found widespread application in the diagnostics of electric arcs, impulse discharges, gas and plasma streams. The characteristics measured in the course of these operations are correlated with the sought local parameters of the object by the Abel equation [1]:

$$2\int_x^R \frac{\varphi(r)\, rdr}{(r^2-x^2)^{1/2}} = f(x), \quad x\in[0,\ R]. \tag{1}$$

Formally, the solution of $\varphi(r)$ can be determined by inverting the Abel equation, i.e.,

$$\varphi(r) = -\frac{1}{\pi}\int_r^R \frac{f'(x)\, dx}{(x^2-r^2)^{1/2}}, \quad r\in[0,\ R], \tag{2}$$

Institute of Thermophysics, Siberian Branch, Academy of Sciences of the USSR. Novosibirsk.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 2, pp. 270-274, August, 1980. Original article submitted January 2, 1980.

0022-0841/80/3902-0869$07.50